Back to Euclidean Geometry: Droz-Farny Demystified

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1. Introduction

The subject of this geometrical excursion is a theorem that has been already the focus of a recent paper here in Mathematical Reflections. Nevertheless, we keep returning to this beautiful theorem of Droz-Farny for its interesting history of complicated proofs that has made it one of the most popular gems in Euclidean Geometry. Its statement is as follows.

THEOREM A (Droz-Farny). Two perpendicular lines are drawn through the orthocenter of a triangle. They intercept a segment on each of the sidelines. The midpoints of these three segments are collinear.

As illustrated in Figure 1, we denote by $A_1, B_1, C_1$, and $A_2, B_2, C_2$ the intersections of the two perpendiculars $d_1, d_2$ with $BC, CA, AB$, respectively. The Droz-Farny theorem states that the midpoints $A_3, B_3, C_3$ of the segments $A_1A_2, B_1B_2, C_1C_2$ are collinear. Arnold Droz-Farny stated this without proof in [2], and, despite the simple configuration, it was only about a century later that the geometry world saw a full proof in...
Sharygin’s classical *Problemas de Geometria*. Years later, on the Hyacinthos forum, the topic was revived and several computational proofs were given by N. Reingold [6], D. Grinberg [4], and M. Stevanovic [8]. In 2004, J.-L. Ayme finally concretized in some sense these proofs by presenting the first synthetic solution (see [1]) - which, although beautiful, did not seem quite natural for it involved some tricky additional constructions. Generalizations and variations around the simple hypothesis have also appeared since then. See for example [3]. The reader is invited to consult Ayme’s paper for a more detailed history and for a brief, yet very interesting, bibliographical note on Arnold Droz-Farny himself.

We will now pass to what we bring new to this picture!

### 2. Main Result

In this section we will prove the following simple result that generalizes Droz-Farny’s theorem and which we think lies at the heart of the theorem itself.

![Diagram of triangle ABC with points A', B', and C']

**Figure 2.**

**Theorem B.** Let $P$ be a point in the plane of a given triangle $ABC$. If $A', B', C'$ are the points where the reflections of lines $PA, PB, PC$ in a given line through $P$ meet the sides $BC, CA, AB$, then $A', B', C'$ are collinear.
Surprise, surprise! This is Problem 5 from the recent USA Mathematical Olympiad (and Problem 6 of the USA Junior Mathematical Olympiad). So, stop for a few seconds and try to figure out how this is related to Droz-Farny!

In any case, we would like to prolong the suspense by passing to the proof of this result rather than offering more details at this point. We first need a basic preliminary result.

**Lemma.** The circumcenter of the triangle determined by the reflections of a point $P$ across the sidelines of a triangle $ABC$ is the isogonal conjugate of the point with respect to triangle $ABC$.

In other words, this point is the intersection of the reflections of $PA$, $PB$, $PC$ in the corresponding internal angle bisectors of triangle $ABC$. The proof is immediate, a simple consequence of the fact that the vertices $A$, $B$, $C$ of the original triangle lie on the line bisectors of the triangle determined by the reflections of $P$ across the sidelines. For more details, the reader is advised to consult for example [5].

Now, we move to the proof itself!

**Proof of Theorem B.** We split it into two parts according to the position of $P$ with respect to the circumcircle of $ABC$.

First, if $P$ lies on the circumcircle, then everything is a simple angle chase! Indeed, since $\angle A'BC' = \angle CBA = \angle CPA = \angle A'PC'$, it follows that points $P$, $A'$, $B$, $C'$ are concyclic, and similarly for $P$, $A$, $B'$, $C'$, and $P$, $A'$, $B'$, $C$. Hence $\angle CA'B' = \angle CPB' = \angle BPC' = \angle BA'C'$, so $A'$, $B'$, $C'$ are collinear.

Now, if $P$ is not on the circumcircle of triangle $ABC$, then let $Q$ be its isogonal conjugate with respect to triangle $ABC$ and let $Q'$ be the isogonal conjugate of $P$ with respect to triangle $AB'C'$. Note that $Q$ is not degenerate - i.e. at infinity - as $P$ does not lie on the circumcircle.
Claim. $Q = Q'$.

Proof of Claim. This represents the key step of the proof. To begin, note that
\[
\angle BQC = \angle BAC + \angle CPB \quad \text{(since $P$ and $Q$ are isogonal conjugates in $ABC$)}
\]
\[
= \angle C'AB' + \angle B'PC'
\]
\[
= \angle C'Q'B' \quad \text{(since $P$ and $Q$ are isogonal conjugates in triangle $AB'C'$)}.
\]

Denote by $X, Y, Z$ the reflections of $P$ in the sides $BC, CA, AB$, and by $X'$ its reflection in the side $B'C'$ of triangle $AB'C'$. Then $\angle ZXY = \angle BQC$ (because $QC$ is orthogonal to $XY$ and $QB$ is orthogonal to $XZ$), whereas $\angle ZX'Y = \angle C'Q'B'$ (because $Q'B'$ is orthogonal to $X'Y$ and $Q'C'$ is orthogonal to $X'Z$). So, since $\angle C'Q'B' = \angle BQC$, we get $\angle ZXY = \angle ZX'Y$. From this it follows that $X, Y, Z, X'$ are concyclic. Nonetheless, the center of the $XYZ$-circle is $Q$, while the center of the $X'Y'Z$-circle is $Q'$ (by the Lemma). Thus, $Q$ and $Q'$ do indeed coincide, proving our Claim.

Next, in a similar way, we can deduce that $Q$ is also the isogonal point of $P$ with respect to triangles $A'BC'$ and $A'B'C$ (by cyclicity). Therefore,
\[
\angle BC'A' = \angle AC'A' = \angle AC'P + \angle PC'Q + \angle QC'A'
\]
\[
= \angle QC'B' + \angle PC'Q + \angle BC'P
\]
\[
= \angle BC'B'
\]
\[
= \angle AC'B'.
\]

This means that $A', B', C'$ are collinear, thus completing the proof. ■

3. Concluding Remarks

Simple enough, right? Other synthetic solutions using Simson’s theorem or even computation approaches via Menelaus are possible. But again, what does this have to do with Droz-Farny’s theorem? Let us take look at the following

Reformulation. Let $\gamma$ and $\delta$ be lines passing through a given point $P$ in the plane of a triangle $ABC$. Let $A', B', C'$ and $A'', B'', C''$ be the intersections of $\gamma$ and $\delta$ with $BC, CA, AB$, respectively. Furthermore, let $X$ be the intersection of $BC$ with the reflection of $AP$ into the internal angle bisector of $\angle A'PA''$, and similarly, define $Y$ and $Z$. Then points $X, Y, Z$ are collinear.

Things might appear brighter now! Of course, this new statement involves the redundant definition of $A', B', C', A'', B'', C''$, but it points out the remarkable fact that Theorem A is nothing but a special case of Theorem B, where we take $P$ to be the orthocenter of $ABC$ and the lines $\gamma$ and $\delta$ to be perpendicular! Why is that precisely? Lines $AP, BP, CP$ in this case, besides being the altitudes of triangle $ABC$, become altitudes in the triangles $A'PA'', B'PB'', C'PC''$, respectively. Hence their reflections in the corresponding internal angle bisectors pass through the circumcenters of triangles $A'PA'', B'PB'', C'PC''$ (as the orthocenter and the circumcenter are isogonal conjugates; see [5]), i.e. they are precisely the midpoints of the interceptions from Droz-Farny!!!

References


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