On The Generalized Ptolemy Theorem

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Introduction. The following note describes a few uses of a relatively less known result in plane geometry, the Generalized Ptolemy Theorem (**GPT**, for short), also known as Casey's Theorem (see Johnson[1]). Featured will be two proofs of the problem proposed by India for the 33rd IMO in Moscow [1993: 255; 1995: 86].

Theorem 1. Circles Ω_1 and Ω_2 are externally tangent at a point I, and both are enclosed by and tangent to a third circle Ω. One common tangent to Ω_1 and Ω_2 meets Ω in B and C, while the common tangent at I meets Ω in A on the same side of BC as I. Then I is the incentre of triangle ABC.

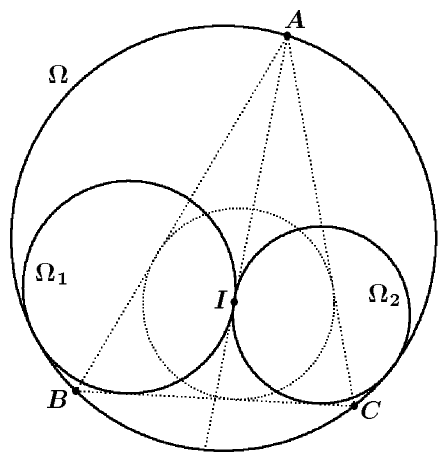

Figure 1.

Proof 1.
Notation: Let t_{ij} refer to the length of the external common tangent to circles i and j (thus the two circles lie on the same side of the tangent). We use the **GPT**, which we state in the following manner.

(**The GPT**) *Let circles α, β, γ, δ all touch the circle Γ, the contacts being all internal or all external and in the cyclical order α, β, γ, δ. Then:*

$$t_{\alpha\beta} \cdot t_{\gamma\delta} + t_{\beta\gamma} \cdot t_{\delta\alpha} = t_{\alpha\gamma} \cdot t_{\beta\delta} .$$

Moreover, a converse also holds: if circles α, β, γ, δ are located such that

$$\pm\, t_{\alpha\beta} \cdot t_{\gamma\delta} \,\pm\, t_{\alpha\delta} \cdot t_{\beta\gamma} \,\pm\, t_{\alpha\gamma} \cdot t_{\beta\delta} = 0$$

for some combination of $+$, $-$ signs, then there exists a circle that touches all four circles, the contacts being all internal or all external.

For a proof of the **GPT** and its converse, please refer to [1].

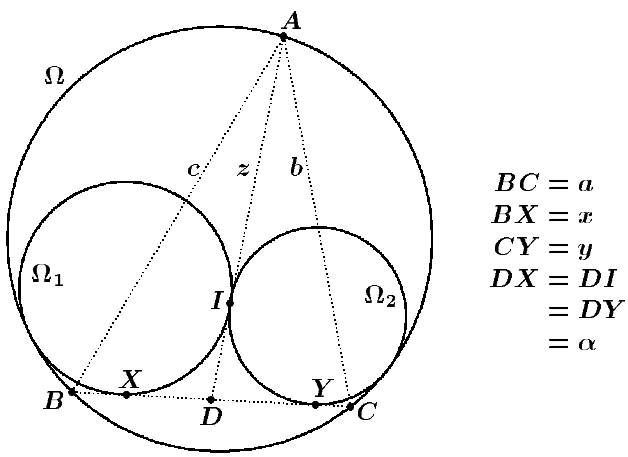

Figure 2.

Consider the configuration shown in Figure 2, where x and y are, respectively, the lengths of the tangents from B and C to Ω_1 and Ω_2; D is $AI \cap BC$; $z = |AI|$; $u = |ID|$; and a, b, c are the sides of $\triangle ABC$.

We apply the **GPT** to the two 4-tuples of circles $(A,\, \Omega_1,\, B,\, C)$ and $(A,\, \Omega_2,\, C,\, B)$. We obtain:

$$az + bx = c(2u + y) \qquad (1)$$

$$az + cy = b(2u + x)\,. \qquad (2)$$

Subtracting (2) from (1) yields $bx - cy = u(c - b)$, so $(x + u)/(y + u) = c/b$, that is, $BD/DC = AB/AC$, which implies that AI bisects $\angle BAC$ and that $BD = ac/(b + c)$. Adding (1) and (2) yields $az = u(b + c)$, so $z/u = (b + c)/a$, that is, $AI/ID = AB/BD$, which implies that BI bisects $\angle ABC$. This proves the result. ■

Proof 2.

Lemma. Let BC be a chord of a circle Γ, and let S_1, S_2 be the two arcs of Γ cut off by BC. Let M be the midpoint of S_2, and consider all possible circles Ω that touch S_1 and BC. Then the length $t_{M\Omega}$ of the tangent from M to Ω is constant for all such Ω. (See Figure 3.)

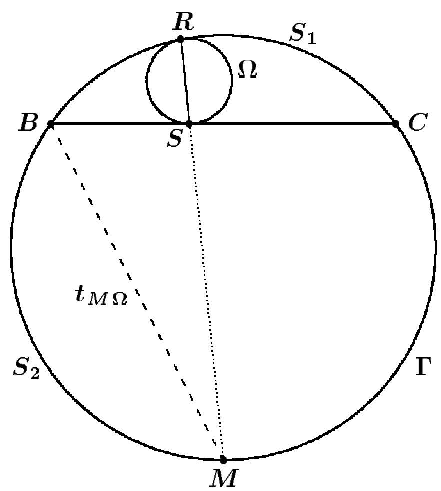

Figure 3.

Proof of Lemma. Let $\Omega \cap \Gamma = R$, $\Omega \cap BC = S$. Applying the **GPT** to the 4-tuple (B, Ω, C, M), we find: $BS \cdot CM + CS \cdot BM = t_{M\Omega} \cdot BC$. Since $BM = CM$, we obtain: $t_{M\Omega} = BM$, a constant. ■

Proof of Theorem 1. (See figure 4.) Let S_1, S_2 be the two arcs of Γ cut off by chord BC, S_1 being the one containing A, and let M denote the midpoint of S_2. Using the above lemma,

$$t_{M\Omega_1} = MB = MI = MC = t_{M\Omega_2}\,.$$

Therefore M has equal powers with respect to Ω_1 and Ω_2 and lies on their radical axis, namely AI. It follows that AI bisects $\angle A$ of $\triangle ABC$.

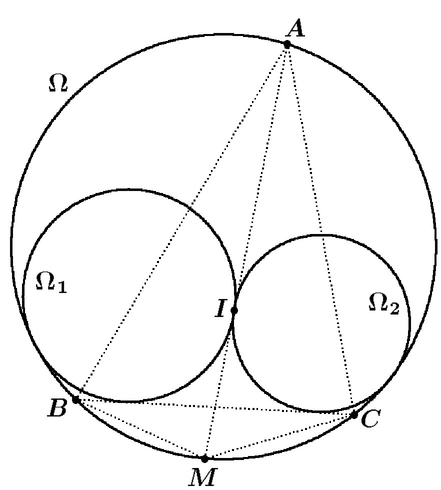

Figure 4.

Next, $\triangle IBM$ is isosceles, so $\angle IBM = \pi/2 - C/2$. Also, $\angle CBM = A/2$, so $\angle IBC = \pi/2 - C/2 - A/2 = B/2$, that is, IB bisects $\angle B$ of $\triangle ABC$. It follows that I is the incentre of $\triangle ABC$. ∎

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For non-believers, here are two more illustrations of the power and economy of the **GPT**.

Theorem 2. Let $\triangle ABC$ have circumcircle Γ, and let Ω be a circle lying within Γ and tangent to it and to the sides AB (at P) and AC (at Q). Then the midpoint of PQ is the incentre of $\triangle ABC$. (See Figure 5.)

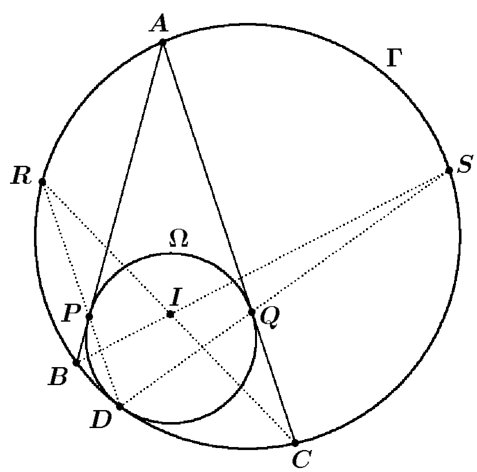

Figure 5.

Proof. Let the **GPT** be applied to the 4-tuple of circles (A, B, Ω, C). Let $AP = x = AQ$. Then:

$$t_{AB} = c,\ \ t_{A\Omega} = AP = x,\ \ t_{AC} = b,$$
$$t_{B\Omega} = BP = c - x,\ \ t_{BC} = a,\ \ t_{\Omega C} = CQ = b - x\,.$$

The **GPT** now gives: $c(b - x) + (c - x)b = ax$, so $x = bc/s$, where $s = (a+b+c)/2$ is the semi-perimeter of $\triangle ABC$. Let I denote the midpoint of PQ; then $IP = x \sin A/2 = (bc/s)\sin A/2$, and the perpendicular distance from I to AB is $IP \cos A/2$, which equals $(bc/s)(\sin A/2)(\cos A/2) = ((1/2)bc \sin A)/s$. But this is just the radius of the incircle of $\triangle ABC$. Since I is equidistant from AB and AC, it follows that I is the incentre of the triangle. ∎

The next illustration concerns one of the most celebrated discoveries in elementary geometry made during the last two centuries.

Theorem 3. (**Feuerbach's Theorem**) The incircle and nine-point circle of a triangle are tangent to one another.

Proof. Let the sides BC, CA, AB of $\triangle ABC$ have midpoints D, E, F respectively, and let Ω be the incircle of the triangle. Let a, b, c be the sides of $\triangle ABC$, and let s be its semi-perimeter. We now consider the 4-tuple of circles (D, E, F, Ω). Here is what we find:

$$\begin{aligned} t_{DE} &= \frac{c}{2}, \quad t_{DF} = \frac{b}{2}, \quad t_{EF} = \frac{a}{2}, \\ t_{D\Omega} &= \left|\frac{a}{2} - (s-b)\right| = \left|\frac{b-c}{2}\right|, \\ t_{E\Omega} &= \left|\frac{b}{2} - (s-c)\right| = \left|\frac{a-c}{2}\right|, \\ t_{F\Omega} &= \left|\frac{c}{2} - (s-a)\right| = \left|\frac{b-a}{2}\right|. \end{aligned}$$

We need to check whether, for some combination of $+$, $-$ signs, we have

$$\pm\, c(b-a) \,\pm\, a(b-c) \,\pm\, b(a-c) = 0.$$

But this is immediate! It follows from the converse to the **GPT** that there exists a circle that touches each of D, E, F and Ω. Since the circle passing through D, E, F is the nine-point circle of the triangle, it follows that Ω and the nine-point circle are tangent to one another. ∎

One would surmise that the **GPT** should provide a neat proof of the following theorem due to Victor Thebault:

> *Let $\triangle ABC$ have circumcircle Γ, let D be a point on BC, and let Ω_1 and Ω_2 be the two circles lying within Γ that are tangent to Γ and also to AD and BC. Then the centres of Ω_1 and Ω_2 are collinear with the incentre of $\triangle ABC$.*

I have, however, not been able to find such a proof, and I leave the problem to the interested reader. We note in passing that Thebault's theorem provides yet another proof of Theorem 1.

References:

[1] R.A. Johnson, *Advanced Euclidean Geometry*, Dover, 1960.

Acknowledgements:

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